Euler's theorem:-

① QN.→ State and prove the Euler's theorem on partial differentiation of homogeneous function of two independent variables. Or,

QN.→ State and prove Euler's theorem for partial differentiation of homogeneous function of two independent variables.

Ans.→ Statement.- If u is a homogeneous function of two independent variables x and y of degree n, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

Proof:-

Let, $u=A_1x^{\alpha_1}y^{\beta_1}+A_2x^{\alpha_2}.y^{\beta_2}+A_3x^{\alpha_3}.y^{\beta_3}+\cdots$ ①

where A_1, A_2, A_3 ------ are constants

i.e. independent of x and y.

Differentiating ① partially w.r.t. x, keeping y constant

$$\frac{\partial u}{\partial x}=A_1\alpha_1x^{\alpha_1-1}y^{\beta_1}+A_2\alpha_2x^{\alpha_2-1}.y^{\beta_2}+A_3\alpha_3x^{\alpha_3-1}.y^{\beta_3}+\cdots$$

multiplying by x, we have,

$$x\frac{\partial u}{\partial x}=A_1\alpha_1x^{\alpha_1}y^{\beta_1}+A_2\alpha_2x^{\alpha_2}y^{\beta_2}+A_3\alpha_3x^{\alpha_3}y^{\beta_3}+\cdots \quad ②$$

Again, diff. ① partially w.r.t. y keeping x constant.

$$\frac{\partial u}{\partial y}=A_1x^{\alpha_1}.\beta_1y^{\beta_1-1}+A_2x^{\alpha_2}.\beta_2y^{\beta_2-1}+A_3x^{\alpha_3}.\beta_3y^{\beta_3-1}+\cdots$$

$$y\frac{\partial u}{\partial y} = A_1 x^{\alpha_1} \beta_1 y^{\beta_1} + A_2 x^{\alpha_2} \beta_2 y^{\beta_2} + A_3 x^{\alpha_3} \beta_3 y^{\beta_3} + \cdots \quad (3)$$

(2) + (3)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = A_1 x^{\alpha_1} y^{\beta_1}(\alpha_1+\beta_1) + A_2 x^{\alpha_2} y^{\beta_2}(\alpha_2+\beta_2) + A_3 x^{\alpha_3} y^{\beta_3}(\alpha_3+\beta_3) + \cdots$$

$$\because \alpha_1+\beta_1 = \alpha_2+\beta_2 = \alpha_3+\beta_3 = \cdots = n$$

$$= A_1 n x^{\alpha_1} y^{\beta_1} + A_2 n x^{\alpha_2} y^{\beta_2} + A_3 n x^{\alpha_3} y^{\beta_3} + \cdots$$

$$= n\left[A_1 x^{\alpha_1} y^{\beta_1} + A_2 x^{\alpha_2} y^{\beta_2} + A_3 x^{\alpha_3} y^{\beta_3} + \cdots\right]$$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$. proved

(2) QN.→ State and prove the Euler's theorm on homogeneous function of three independent variables.

or, State and prove Euler's theorm on partial differentiation for three variables.

Ans.→ Statement — If u is a homogeneous function of three independent variables x, y and z of degree n^{th} then.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$$

Proof :—

Let $u = A_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + A_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + A_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \cdots \quad (1)$

where, $A_1, A_2, A_3 \cdots$ are constants i.e. independent of x, y and z.

Diff. ① partially w.r.t. x keeping y and z constants, we get

$$\frac{\partial u}{\partial x} = A_1\alpha_1 x^{\alpha_1-1} y^{\beta_1} z^{\gamma_1} + A_2\alpha_2 x^{\alpha_2-1} y^{\beta_2} z^{\gamma_2} + A_3\alpha_3 x^{\alpha_3-1} y^{\beta_3} z^{\gamma_3} + \cdots$$

multiplying by x, we have,

$$x\frac{\partial u}{\partial x} = A_1\alpha_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + A_2\alpha_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + A_3\alpha_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \cdots \quad (2)$$

Again, diff. ① partially w.r.t. y keeping x and z as constants, we get,

$$\frac{\partial u}{\partial y} = A_1\beta_1 x^{\alpha_1} y^{\beta_1-1} z^{\gamma_1} + A_2\beta_2 x^{\alpha_2} y^{\beta_2-1} z^{\gamma_2} + A_3\beta_3 x^{\alpha_3} y^{\beta_3-1} z^{\gamma_3} + \cdots$$

multiplying by y, we have,

$$y\frac{\partial u}{\partial y} = A_1\beta_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + A_2\beta_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + A_3\beta_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \cdots \quad (3)$$

Again, diff. ① partially w.r.t. keeping x and y as constants, we get,

$$\frac{\partial u}{\partial z} = A_1\gamma_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1-1} + A_2\gamma_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2-1} + A_3\gamma_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3-1} + \cdots$$

multiplying by z, we have,

$$z\frac{\partial u}{\partial z} = A_1\gamma_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + A_2\gamma_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + A_3\gamma_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \cdots \quad (4)$$

② + ③ + ④, we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = A_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1}(\alpha_1+\beta_1+\gamma_1) + A_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2}(\alpha_2+\beta_2+\gamma_2) + A_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3}(\alpha_3+\beta_3+\gamma_3) + \cdots$$

$$\therefore \alpha_1+\beta_1+\gamma_1 = \alpha_2+\beta_2+\gamma_2 = \alpha_3+\beta_3+\gamma_3 = \cdots\cdots = n.$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = A_1 n x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + A_2 n x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + A_3 n x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \cdots$$

$$= n\left[A_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + A_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + A_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3} + \cdots\right]$$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = nu.$$ Proved.

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